Nodal and Periastron Precession of Inclined Orbits in the Field of a Rapidly Rotating Neutron Star .

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Abstract

We derive a formula for the nodal precession frequency and the Keplerian period of a particle at an arbitrarily inclined orbit (with a minimum latitudinal angle reached at the orbit) in the post-Newtonian approximation in the external field of an oblate rotating neutron star (NS). We also derive formulas for the nodal precession and periastron rotation frequencies of slightly inclined low-eccentricity orbits in the field of a rapidly rotating NS in the form of asymptotic expansions whose first terms are given by the Okazaki–Kato formulas. The NS gravitational field is described by the exact solution of the Einstein equation that includes the NS quadrupole moment induced by rapid rotation. Convenient asymptotic formulas are given for the metric coefficients of the corresponding space-time in the form of Kerr metric perturbations in Boyer–Lindquist coordinates.

Key words: neutron stars, luminosity, disk accretion, X-ray radiation.

INTRODUCTION

The X-ray flux from low-mass X-ray binaries (LMXBs) commonly exhibits two peaks in the power spectrum at frequencies ~ 1 kHz and one peak in the range 10–100 Hz (Van der Klis 2000). Stella and Vietri (1998) proposed to interpret the frequency difference between the kHz peaks as the periastron rotation frequency of low-eccentricity orbits: the observed decrease in this difference for some LMXBs was explained by a radiating clump approaching the marginally stable orbit at which the periastron rotation frequency becomes zero. The low-frequency peak was interpreted as the precession frequency of circular Keplerian orbits inclined to the equator (Merloni *et al.* 1998; Morsink and Stella 1999; Stella and Vietri 1999; Psaltis *et al.* 1999). The formulas of Okazaki *et al.* (1987) derived in terms of the Kerr solution for orbits slightly inclined to the equatorial plane are commonly used to compare models with observations. The importance of the precession of inclined orbits in the fields of rotating black holes in interpreting the quasi-periodic oscillations (QPOs) of the X-ray flux

from pulsars and black-hole candidates was first pointed out by Cui *et al.* (1998). Inclined orbits in Kerr and Kerr–Newman fields were first considered by Wilkins (1972) (uncharged case) and Johnston and Ruffini (1974) (for a charged rotating black hole). Van Kerkwijk *et al.* (1998) explained the puzzling spindown and spinup of some X-ray pulsars by the fact that the accretion-disk tilt to the equatorial plane in inner regions can become larger than 90°! Previously (Sibgatullin 2001), we derived analytic expressions for the nodal precession and periastron rotation frequencies for orbits *arbitrarily inclined* to the equatorial plane[1]. The nodal precession of the *marginally stable orbits* is described as a function of the corresponding Keplerian frequency for various inclinations of these orbits. For a Keplerian frequency of 1200 Hz and a NS mass of 2.2 $M_\odot$, the nodal frequency in the marginally stable orbit was shown to change from 41 Hz to 123 Hz as the inclination to the equator changes from 0 to 90°, i.e., by a factor of 3!

However, the results for black holes need to be significantly corrected for neutron stars, because a rapidly rotating NS becomes oblate and a quadrupole moment appears. For orbits with low inclinations to the equatorial plane, Markovic (2000) derived formulas for the nodal and periastron precession frequencies in the post-Newtonian approximation by taking into account the NS quadrupole moment. Morsink and Stella (1999) and Vietri and Morsink (1999) *numerically* calculated ν_r and $\nu_{\rm nod}$ as functions of ν_ϕ for low inclinations and low eccentricities for various NS equations of state.

Here, our goal is to derive *analytic* formulas for ν_r and $\nu_{\rm nod}$ for *low* inclinations and low eccentricities in the form of asymptotic expansions, which transform to the formulas of Okazaki *et al.* (1987) and Kato (1990) at a zero quadrupole moment. The exact quadrupole solution extracted from the more general solution found by Man'ko *et al.* (1994) formed a basis for our study. This solution was reduced to the simplest form in Weyl coordinates. For clarity, the exact quadrupole solution is also given asymptotically as a Kerr metric perturbation in Boyer–Lindquist coordinates.

Sibgatullin and Sunyaev (1998, 2000a, 2000b) (below referred to as SS 98, SS 00a, and SS 00b) provided formulas for the luminosity and spindown rate for various NS equations of state. These authors proposed a method of analytically constructing the quadrupole moment b as a function of the Kerr parameter j and NS rest mass m, $b = b(j, m)$, based on the main NS thermodynamic function — its gravitational mass M as a function of the Kerr parameter and rest mass $M = M(j, m)$. In this case, the space-time geometry for the exact quadrupole solution plays a crucial role in finding the marginally stable orbit. The dependence $b(j, m)$ was constructed using numerical data from Cook *et al.* (1994) and the numerical code by Stergioulas (1998) for the marginally stable orbit. Laarakkers and Poisson (1998) found a parabolic dependence of the quadrupole moment on the Kerr parameter at $j << 1$ by using direct calculations from the formula of Ryan (1995b, 1997) for several equations of state. The physical parameters of the marginally stable orbit were studied by Shibata and M. Sasaki (1998) using expansions at large radii.

An important outstanding question (which is not considered here) is the passage from a viscous accretion disk (Shakura and Sunyaev 1973, 1976) to free particles near the marginally stable orbit. Another complicated question (which is not considered here either) is the behavior of bound trajectories at the marginally stable orbit for their finite inclination to the equatorial plane.

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[1]To be more precise, with an arbitrary minimum latitudinal angle, which can be reached in a bound trajectory.

1 THE EXTERNAL GRAVITATIONAL FIELDS OF ROTATING NEUTRON STARS

The efficiency of the exact quadrupole solution in describing the external fields of rapidly rotating neutron stars with masses larger than 1 $M_\odot$ was demonstrated in SS 98, SS 00a, and SS 00b. In contrast to the Kerr solution, this solution contains an additional constant, which has the meaning of intrinsic (non-Kerr) NS quadrupole moment, and is symmetric relative to the equatorial plane. The solution under discussion is contained as a special case in the five-parameter (mass, angular momentum, quadrupole moment, electric charge, and magnetic dipole) solutions obtained by Man'ko *et al.* (1994) using the method developed in our book (Singatullin 1991)[2]. However, the expressions for the metric coefficients and the 4-potential of the electromagnetic field are cumbersome. To solve problems with disk accretion in the equatorial plane, we used simple expressions for the metric of the quadrupole solution in the equatorial plane (SS 98, SS 00a). Since the importance of the quadrupole solution in describing the fields of rotating NS gravitational fields is beyond question, we give here the corresponding exact solution of the Einstein equations in the entire space [3]. Thus, the metric coefficients in the square of the interval in vacuum[4],

$$ds^2 = -f(dt - \omega d\phi)^2 + \frac{\rho^2}{f}d\phi^2 +$$
$$+ \frac{\exp 2\gamma}{f}(d\rho^2 + dz^2) \tag{1}$$

in the exact quadrupole solution are[5]

$$f = \text{Re}\left(\frac{A - B}{A + B}\right), \tag{2}$$
$$f\omega = 2\text{Re}\left(\frac{j(A - B) + iC}{A + B}\right),$$
$$e^{e\gamma} = \frac{A\bar{A} - B\bar{B}}{16k_+^4 k_-^4 R_+ R_- r_+ r_-}.$$

In formulas (2), we use the following notation

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[2]For the relationship of this method to the soliton solutions of Kramer and Neugebauer (1980), see Ernst (1994) and Man'ko and Ruiz (1998).

[3]The metric of the exact solutions for the Einstein equations with a finite set of multipole moments is given in general form in SS 00. The method of constructing exact asymptotic flat solutions for the Einstein-Maxwell set of equations with data specified on the symmetry axis was developed in our book (Sibgatullin 1991). The entire problem reduces to solving the only homogeneous singular equation with the Cauchy kernel on a segment with an additional normalization condition.

[4]in Papapetru form.

[5]Here, we use a system of units with $M = c = G = 1$.

$$A = k_-^2(R_- + r_+)(R_+ + r_-) - \tag{3}$$
$$- k_+^2(R_- - r_-)(R_+ - r_+),$$
$$B = k_+k_-((k_+ + k_-)(r_+ + r_-) +$$
$$+ (k_- - k_+)(R_- + R_+)),$$
$$C = zB + k_+k_-(k_-(R_+r_+ - R_-r_-) +$$
$$+ k_+(R_+r_- - R_-r_+) + 2b(R_+ - R_- + r_- - r_+)),$$
$$R_\pm = \sqrt{\rho^2 + (z \pm (k_+ + k_-)/2)^2}(\sqrt{1 - j^2} \pm ij), \tag{4}$$
$$r_\pm = \sqrt{\rho^2 + (z \pm (k_+ - k_-)/2)^2}(\sqrt{1 - j^2} \pm ij),$$
$$k_- = \sqrt{1 - j^2 - 4b}, \quad k_+ = \sqrt{1 - j^2}.$$

The metric coefficient ω becomes zero on the symmetry axis at $|z| > (k_+ + k_-)/2$. We assume that ω on the symmetry axis becomes zero everywhere outside the rotating NS (the condition for the absence of conical points). Solution (2)–(4) was obtained for the Ernst potential on the symmetry axis:

$$\mathbf{E} = \frac{A - B}{A + B}\Big|_{\rho=0} = \frac{z^2 + (ij - 1)z + b^2}{z^2 + (ij + 1)z + b^2}.$$

Note that $(j(A - B) + iC)/(A + B) = i$ on the symmetry axis.

In the special case where $b = (1 - j^2)/4$, the constant k_- becomes zero. In this case, it is convenient to pass to the coordinates $\rho = \sin\theta\sqrt{(r-1)^2 - (1-j^2)/4}$, $z = (r - 1)\cos\theta$. Passing to the limit $k_- \to 0$, the metric coefficients in the square of the interval

$$ds^2 = -f(dt - \omega d\phi)^2 + \frac{\rho^2}{f}d\phi^2 + \tag{5}$$
$$+ \frac{g}{f}\left(\frac{dr^2}{(r-1)^2 - (1-j^2)/4} + d\theta^2\right)$$

may be represented as

$$f = \mathrm{Re}\left(\frac{a}{b}\right), \quad f\omega = 2\mathrm{Re}\left(\frac{c}{b}\right), \tag{6}$$
$$g = \frac{\mathrm{Re}(ab^*)}{4(4(r-1)^2 - (1-j^2)m^2)^3}.$$

Here, the asterisk denotes a complex conjugate and the following notation is used with $m \equiv \cos\theta$:

$$a = -27 - 5j^2 + 18ijm - 2ij^3m - 6ijm^3$$
$$+ 6ij^3m^3 - j^2m^4 + j^4m^4 + 108r + 4j^2r -$$
$$- 48ijmr + 4ijm^3r - 4ij^3m^3r - 144r^2 +$$
$$+ 48ijmr^2 + 80r^3 - 16ijmr^3 - 16r^4,$$
$$b = -3 + 3j^2 + 14ijm + 2ij^3m - 2ijm^3 +$$
$$+ 2ijm^3 - j^2m^4 + j^4m^4 + 10r - 4j^2r -$$
$$- 48ijmr + 4ijm^3r - 4ij^3m^3r - 48r^2 +$$
$$+ 48ijmr^2 + 48r^3 - 16ijmr^3 - 16r^4,$$
$$c = ij^4m(-1 + 2m^2) - im(-3 + 2r)(-1 + 2r)^3 -$$
$$- 2ij^2m(-2 + m^2 + 2r) + j(-13 + m^4(1 - 2r) +$$
$$+ 46r - 48r^2 + 16r^3) + j^3(-3 + 2r + m^4(-1 + 2r)).$$

Let us again turn back to the general exact quadrupole solution (2)–(4) and pass to Boyer–Lindquist coordinates $\rho = \sin\theta \times \sqrt{(r-1)^2 - (1-j^2)}$, $z = (r-1)\cos\theta$.

Let us represent the square of the interval as

$$ds^2 = -fdt^2 + 2f\omega dt d\phi + \tag{7}$$
$$+ \sin^2\theta \left(\frac{r^2 - 2r + j^2}{f} - f\omega^2 \right) d\phi^2 +$$
$$+ \frac{\Gamma}{f} \left(\frac{dr^2}{r^2 - 2r + j^2} + d\theta^2 \right),$$

where

$$f = \frac{F}{r^2 + j^2\cos^2\theta}, \quad \omega = \frac{W}{r^2 + j^2\cos^2\theta}. \tag{8}$$

We will seek the functions F, Γ, and W using the exact solution (2)–(4) in the form of series in inverse powers of the radius. At $b = 0$, the series for F, Γ, and W break off on the second or third term, giving the exact Kerr solution

$$F_k = \Gamma_k = r^2 - 2r + j^2\cos^2\theta, \quad W_k = -2jr\sin^2\theta.$$

At $b \neq 0$, the series for F, G, and W contain an infinite number of terms and diverge on the event horizon. This fact is related to the so-called *no hair theorem*[6] (see Misner *et al.* (1973), Novikov and Frolov (1986), Sibgatullin (1984) for a more detailed discussion of the black-hole theory). A collapsing star loses its magnetic and quadrupole moment and its field asymptotically approaches the field of a rotating black hole (Ginzburg and Ozernoi 1964; Doroshkevich *et al.* 1965).

Cumbersome transformations (in these formulas, $m \equiv \cos\theta$) yielded

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[6] According to Wheeler's figurative expression, "a black hole has no hair ".

$$F = r^2 - 2r + j^2 m^2 + b\left(\frac{3m^2 - 1}{r^2}(r+1)\right) \tag{9}$$

$$\frac{1}{r^3}\left(\frac{3}{4}(1+j^2+b) - \frac{1}{2}(3+25j^2+15b)m^2\right.$$

$$\left. + \frac{1}{4}(-5+63j^2+35b)m^4\right) + \mathrm{o}\left[\frac{1}{r^4}\right]\right);$$

$$\Gamma = r^2 - 2r + j^2 m^2 + b(1-m^2)$$

$$\times \left(\frac{3}{2r^3}(5m^2-1)(r+2) + \frac{1}{4r^4}(-17-j^2\right.$$

$$+ 8b + 2(35+10j^2+56b)m^2 + (35-$$

$$\left. -35j^2 - 180b)m^4\right) + \mathrm{o}\left[\frac{1}{r^5}\right]\right);$$

$$W = 2j(1-m^2)\left(-r + b\left(\frac{1}{4r^3}(5m^2-1)\times\right.\right.$$

$$(4r+7) + \frac{1}{8r^4}(-24-b+2(54+20j^2+63b)m^2$$

$$\left.\left. + (12-72j^2-189b)m^4\right) + \mathrm{o}\left[\frac{1}{r^5}\right]\right)\right).$$

2 PERIASTRON AND NODAL PRECESSION OF ORBITS WITH LOW INCLINATIONS TO THE EQUATORIAL PLANE AND WITH LOW ECCENTRICITIES

The eikonal equation $g^{ij}S_{,i}S_{,j} = -1$ in stationary spaces with axial symmetry and with the square of the interval (1) have solutions of the form $S = -Et + L\phi + \tilde{S}(\rho, z)$, where $\tilde{S}$ satisfies the equation

$$(\tilde{S}_{,\rho}^2 + \tilde{S}_{,z}^2)fe^{-2\gamma} = V(\rho, z) \equiv \frac{E^2}{f} - \tag{10}$$

$$- \frac{f}{\rho^2}(L - \omega E)^2 - 1.$$

Consider nearly circular orbits in the equatorial plane. For the latter, $V = V_{,z} = V_{,\rho} = 0$ at $z = 0$. Consequently, expanding the right-hand part of Eq. (10) for perturbations in a Taylor series to within quadratic terms, we have

$$(\tilde{S}_{,\rho}^2 + \tilde{S}_{,z}^2)fe^{-2\gamma} = \frac{1}{2}(V_{,\rho\rho}(\Delta\rho)^2 + V_{,zz}z^2) + \text{const.} \tag{11}$$

The constant in Eq. (11) is related to the perturbation of V when the constants E and L are perturbed for nonequatorial, noncircular orbits. Equation (11) can be solved by the

separation of variables, $\tilde{S} = S_1(\Delta\rho) + S_2(z)$. Consider the Hamiltonian system associated with Eq. (11). We will seek $\Delta\rho$ and z in the form $\Delta\rho = \epsilon_1 \sin\xi$, $z = \epsilon_2 \sin\zeta$. We then obtain

$$\frac{d\zeta}{ds} = \sqrt{-fe^{-2\gamma}V_{zz}/2}, \quad \frac{d\xi}{ds} = \sqrt{-fe^{-2\gamma}V_{\rho\rho}/2}, \tag{12}$$
$$\frac{d\phi}{ds} = \frac{f}{\rho^2}(L - \omega E), \quad \frac{dt}{ds} = \frac{E}{f} + \frac{f\omega}{\rho^2}(L - \omega E).$$

Note that the right-hand parts of Eqs. (12) are constant.

Let us now make use of the expression for the energy and angular momentum of particles in equatorial circular orbits (SS 98, SS 00a):

$$E = \frac{\sqrt{f}}{\sqrt{1 - f^2 p^2/r}}, \quad L - \omega E = pE, \tag{13}$$

$$p \equiv \rho^2(-\lambda + \sqrt{\lambda^2 + \mu - \mu^2 r})/n,$$
$$\lambda \equiv f\dot{\omega}, \mu \equiv \dot{f}/f, n \equiv f - r\dot{f}.$$

Here, the dot denotes a derivative with respect to ρ^2.

For the angular velocity of a particle in a Keplerian equatorial circular orbit, the following formula can be derived from Eqs. (12):

$$\omega_\phi = \frac{p}{(\rho^2/f^2 + \omega p)}.$$

Below, it is convenient to introduce a new quantity instead of p: $\tau \equiv \rho^2/(fp)$. The formula for the angular velocity can then be rewritten as

$$\nu_\phi = \frac{1}{T}, \quad T/(2\pi) = \omega + \tau/f, \tag{14}$$
$$\tau \equiv (\lambda + \sqrt{\lambda^2 + \mu - \mu^2 r})/\mu.$$

For the rotation frequency from periastron to periastron, $2\pi\nu_r = d\xi/dt$, we obtain from Eqs. (12)

$$\nu_r^2 = \nu_\phi^2 M, \tag{15}$$

where we use the notation

$$M \equiv 2e^{-2\gamma}\left(-gf\left(\frac{1}{f}\right)_{,qq}\tau^2 + \frac{q^3}{f}\left(\frac{f}{q}\right)_{,qq} - \right.$$
$$\left. - 4q^2\left(\frac{f}{q}\right)_{,q}\tau\omega_{,q} - 2q\tau f\omega_{qq} + 2\tau^2 f^2(\omega_{,q})^2\right)|_{z=0},$$

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$q \equiv \rho^2$. In exactly the same way, for the rotation frequency of the maximum rise in z, $2\pi\nu_\theta = d\zeta/dt$, we obtain from Eqs. (12) using the explicit expression (10) for V

$$\nu_\theta^2 = \nu_\phi^2 N, \tag{16}$$

$$N = \frac{e^{-2\gamma}}{f}(f_{,zz}(\rho^2 + \tau^2) - 2f\omega_{zz}\tau)|_{z=0}.$$

For the periastron and nodal precession frequencies, we have

$$\nu_{\mathrm{per}} = \nu_\phi(1 - \sqrt{M}), \quad \nu_{\mathrm{nod}} = \nu_\phi(1 - \sqrt{N}). \tag{17}$$

We will seek the functions $T(r)$, $N(r)$, and $M(r)$ in coordinates (7) in the form of expansions in terms of inverse powers of r. At $b = 0$, the series break off to give the exact formulas of Okazaki *et al.* (1987) and Kato (1990) for the Kerr metric

$$M = 1 - \frac{6}{r} + \frac{8j}{r^3/2} - \frac{3j^2}{r^2}, \quad N = 1 - \frac{4j}{r^{3/2}} + \frac{3j^2}{r^2},$$

$$T/(2\pi) = j + r^{3/2}.$$

Let $b \neq 0$; using coordinates (7), we obtain

$$M = 1 - \frac{6}{r} + \frac{8j}{r^{3/2}} - \frac{3(j^2 + b)}{r^2} + 3b\left(-\frac{5}{r^3}\right. \tag{18}$$

$$+ \frac{8j}{r^{7/2}} - \frac{1}{2r^4}(16 + 7j^2 + 2b) + \frac{12j}{r^{9/2}} -$$

$$\left. -\frac{1}{4r^5}(51 + 3j^2 + 31b) + \mathrm{o}\left[\frac{1}{r^{11/2}}\right]\right);$$

$$N = 1 - \frac{4j}{r^{3/2}} + \frac{3(j^2 + b)}{r^2} + 3b\left(\frac{2}{r^3} - \right. \tag{19}$$

$$\left. -\frac{5j}{r^{7/2}} + \frac{1}{2r^4}(8 + 7j^2 + 2b) - \frac{9j}{r^{9/2}} + \mathrm{o}\left[\frac{1}{r^5}\right]\right);$$

$$T/(2\pi) = r^{3/2} + j + b\left(-\frac{3}{4r^{1/2}} - \frac{5}{4r^{3/2}}\right. \tag{20}$$

$$+ \frac{3j}{2r^2} - \frac{3}{32r^{5/2}}(10 + 10j^2 + b) + \frac{5j}{r^3} -$$

$$\left. -\frac{3}{16r^{7/2}}(29 + 13j^2 + 4b) + \mathrm{o}\left[\frac{1}{r^4}\right]\right).$$

Using Eqs. (19) and (20) for the nodal velocity of slightly inclined orbits, we derive a Taylor series in inverse powers of the radius:

$$2\pi\nu_{\mathrm{nod}} = \frac{2j}{r^3} - \frac{3(b + j^2)}{2r^{7/2}} - \frac{3b}{r^{9/2}} + \frac{3j}{2r^5}(-j^2 + 5b) - \tag{21}$$

$$-\frac{3}{8r^{11/2}}(16b - 3j^4 + 11j^2b + 4b^2) + \mathrm{o}\left[\frac{1}{r^6}\right].$$

At $b = 0$, we obtain a Taylor expansion of the formula by Okazaki *et al.* (1987) and Kato (1990) from Eq. (21).

For orbits with low inclinations to the equatorial plane, Markovic (2000) derived formulas for the periastron and nodal precession frequencies in the *post-Newtonian approximation.* His formulas for the nodal precession frequency is equivalent to Eq. (21) if the first two terms are retained in it. The latter are the sum of the Newtonian precession of a slightly inclined orbit in a gravitational field with a quadrupole moment and the precession of Lense and Thirring (1918).

Thus, for orbits with low inclinations to the equatorial plane and with low eccentricities, allowance for NS oblateness in Eqs. (17) for the nodal and periastron precession of orbits with rapid rotation gives a **large contribution for the hard equations of state**. The method for constructing the dependence of quadrupole moment on the NS rest mass and Kerr parameter is described in SS 00a; for the equations of state A and FPS, specific functions $b(j, m)$ were constructed at rest masses larger than the solar mass and smaller than the critical mass according to the static stability criterion. Laarakkers and Poisson (1998) found an almost quadratic dependence of quadrupole moment on the Kerr parameter at small j. In SS 98, the dependence $b(j)$ was studied for normal sequences with $1.4 M_\odot$ in the static limit and for normal sequences unstable in the static limit *over the entire range of Kerr parameters* in which the equatorial rotation velocity on the stellar surface is lower than the equatorial Keplerian velocity.

3 NODAL PRECESSION OF ORBITS WITH ARBITRARY INCLINATIONS TO THE EQUATORIAL PLANE IN THE POST-NEWTONIAN APPROXIMATION

Consider the non-Kerr terms in Eqs. (9) as small Kerr metric perturbations: $F = F_k(1 + v)$; $\Gamma = \Gamma_k(1 + z)$; $W = W_k(1 + w)$. The eikonal equation (10) in Boyer–Lindquist coordinates can be written as

$$(\tilde{S}_{,r}^2 \Delta + \tilde{S}_{,\theta}^2)(1 + v - z) = (E(r^2 + j^2) - L_j)^2/\Delta - \tag{22}$$
$$- (L - jE)^2 + \cos^2\theta\left((E^2 - 1)j^2 - \frac{L^2}{\sin^2\theta}\right) -$$
$$- v\left(\left(\frac{(r^2 + j^2)^2}{\Delta} - j^2\sin^2\theta\right)E^2 + \frac{L^2}{\sin^2\theta} -\right.$$
$$\left. - \frac{L^2 j^2}{\Delta}\right) - \frac{4w(2j^2 r^2 \sin^2\theta + ELjr)}{\Delta}.$$

The asymptotic result for the nodal precession frequency that follows from Eq. (22) can also be obtained as follows.

Consider the energy integral for a particle moving in the gravitational field of a point mass with a quadrupole moment in the Newtonian approximation:

$$\frac{1}{2}\left(r^2\left(\frac{d\theta}{dt}\right)^2 + \frac{L^2}{r^2\sin^2\theta} + \left(\frac{dr}{dt}\right)^2\right) - \frac{1}{r}$$

$$+ \frac{b}{2r^3}(3\cos^2\theta - 1) = H, \quad H = \text{const.} \tag{23}$$

In the absence of quadrupole moment for circular orbits, we have

$$H = -1/2; \quad p_r = 0; \quad L^2 = r\sin^2 s; \tag{24}$$
$$\cos\theta = \cos s \cos(t/r^{3/2}).$$

We restrict our analysis to the case $b/r^2 << 1$ and denote $\tau \equiv t/r^{3/2}$. The energy integral, to within quantities of the second order of smallness in b/r^2, can be written as

$$\left(\frac{d\theta}{d\tau}\right)^2 = \left(1 - \frac{L^2}{r\sin^2\theta}\right)\left(1 - 4\frac{\delta r}{r}\right) \tag{25}$$

$$+ 2r\delta H - \frac{b}{r^2}(3\cos^2\theta - 1).$$

We write the constant H via its value at the turning point $d\theta/d\tau = 0$:

$$1 + rH = \frac{L^2}{r\sin^2 s} + \frac{b}{2r^2}(3\cos^2 s - 1). \tag{26}$$

With (26), the energy integral (25) takes the form

$$\left(\frac{d\theta}{d\tau}\right)^2 = \frac{\sin^2\theta - \sin^2 s}{\sin^2\theta} \tag{27}$$

$$\times \left(\frac{L^2}{r\sin^2 s} - 4\frac{\delta r}{r} + \frac{3b}{r^2}\sin^2\theta\right).$$

An analog of circular orbits in a quadrupole field are perturbed orbits with a zero mean deviation δr from a sphere of constant radius:

$$\int \delta r \, d\phi(t) = 0. \tag{28}$$

In Eq. (28), the integral is taken over the meridional period T_θ in which the particle again returns to the minimum angle s. Given (28), the azimuthal angle ϕ changes in time $T_\theta/4$ by

$$\Delta\phi = \int_s^{\pi/2} \sqrt{\frac{L^2}{r\sin^2 s}} \frac{d\theta}{\sin^2\theta}\left((\cot^2 s - \cot^2\theta)\times\right. \tag{29}$$

$$\times\left.\left(\frac{L^2}{r\sin^2 s} + \frac{3b}{r^2}\sin^2\theta\right)\right)^{1/2} \approx \frac{\pi}{2}\left(1 - \frac{3b}{2r^2}\sin s\right).$$

Consequently, the nodal precession because of NS oblateness (the presence of a quadrupole moment) is

$$2\pi\nu_{\mathrm{nod}} = -\frac{3}{2r^{7/2}}b\sin s.$$

Combining this result with the formula for the precession frequency of inclined circular orbits in a Kerr field [see Eq. (52) in Sibgatullin (2001)], we **derive the sought-for formula for the nodal precession frequency in the post-Newtonian approximation**:

$$2\pi\nu_{\mathrm{nod}} = -\frac{2j}{r^3} - \frac{3}{2r^{7/2}}(b+j^2)\sin s. \tag{30}$$

Formula (30) at $s = \pi/2$ transforms to the formula of Markovic (2000). In exactly the same way, for the azimuthal (Keplerian) period at an inclined orbit in the field of an oblate star in the post-Newtonian approximation, we obtain using (28)

$$T/(2\pi) = r^{3/2} + (3\sin s - 2)j - \frac{3}{4}\frac{b}{r^{1/2}}(2 - \sin s) + \frac{9}{4}\frac{1}{r^{1/2}}\cos^2 s\,(j^2 + b). \tag{31}$$

At $s \to \pi/2$, Eq. (31) matches Eq. (20) in the principal terms.

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